Faculty of Science, Technology, Engineering and Mathematics
M337 Complex analysis

M337

TMA 02 2020J

Covers Book B **Cut-off date 10 February 2021**

You can submit this TMA either by post to your tutor or electronically as a PDF file by using the University's online TMA/EMA service.

Before starting work on it, please read the document *Student guidance for preparing and submitting TMAs*, available from the 'Assessment' area of the M337 website.

Your work should be written in good mathematical style, as demonstrated by the example and exercise solutions in the study units. You should explain your solutions carefully, using appropriate notation and terminology, and write in sentences. As usual, you should simplify algebraic answers where possible.

In the wording of the questions:

- *write down* or *state* means 'write down without justification'
- *find*, *determine*, *calculate*, *explain*, *derive*, *evaluate* or *solve* means that we require you to show all your working in giving an answer
- *prove*, *show* or *deduce* means that you should carefully justify each step of your solution.

Make sure to reference any significant result from the module materials that you use, and check that all the conditions of the result are satisfied.

Question 1 (Unit B1) – 25 marks

(a) (i) Evaluate $\displaystyle\int_{\Gamma_1} i\,\overline{z}\,dz$, where $\Gamma_1 : \gamma_1(t) = e^{it}$ $(t \in [-\pi, \pi/2])$. [4]

(ii) Evaluate $\displaystyle\int_{\Gamma_2} i\,\overline{z}\,dz$, where $\Gamma_2 : \gamma_2(t) = -1 + (1+i)t$ $(t \in [0,1])$. [4]

(iii) Use your answers to parts (a)(i) and (a)(ii) to prove that the function $f(z) = i\,\overline{z}$ cannot be the derivative of an entire function. [2]

(b) Evaluate the following integrals, where Γ is any contour from $-i$ to i. In each case you should give your answer in Cartesian form and justify your method fully.

(i) $\displaystyle\int_{\Gamma} z\sinh(z^2)\,dz$ [4]

(ii) $\displaystyle\int_{\Gamma} z\sinh z\,dz$ [4]

(c) Use the Estimation Theorem to find an upper estimate for the modulus of the integral

$$\int_{\Gamma} \frac{ze^z - 2z^3}{(z^2+1)(z-4)}\,dz,$$

where Γ is the upper half of the circle $\{z : |z| = 2\}$. [7]

Question 2 (Unit B2) – 25 marks

(a) Evaluate the integral

$$\int_C \frac{\cos \pi z}{z(z-2)^2}\,dz$$

when C is each of the following circles.

(i) $C = \{z : |z - 2i| = 1\}$ [3]

(ii) $C = \{z : |z| = 1\}$ [4]

(iii) $C = \{z : |z| = 3\}$ [6]

(b) Suppose that f is an entire function and K is a positive constant such that

$$|f(z)| \leq K|z|, \quad \text{for all } z \in \mathbb{C}.$$

(i) Let $\alpha \in \mathbb{C}$ and $R > 0$. Prove that if $|z| = R$, then

$$\left|\frac{f(z)}{(z-\alpha)^2}\right| \leq \frac{RK}{(R - |\alpha|)^2}.$$ [3]

(ii) Use Cauchy's First Derivative Formula, the Estimation Theorem and part (b)(i) to prove that if $\alpha \in \mathbb{C}$ and $R = 2|\alpha| + 1$, then

$$|f'(\alpha)| \leq \frac{R^2 K}{(R - |\alpha|)^2} \leq 4K.$$ [6]

(iii) Use Liouville's Theorem to deduce from part (b)(ii) that $f(z) = cz$, for some constant c. [3]

Question 3 (Unit B3) – 25 marks

(a) Determine the disc of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}(z-2i)^n.$$ [3]

(b) Let $f(z) = \operatorname{Log}(2i - z)$.

Use Taylor's Theorem to determine the Taylor series about i for f, up to the term in $(z-i)^2$, and find an open disc centred at i on which the series converges. [5]

(c) Find the Taylor series about 0 for each of the following functions.

(i) $f(z) = e^{2z}(1 + \sin z)$ (up to the term in z^3) [4]

(ii) $f(z) = \operatorname{Log}(e^z + \sin z)$ (up to the term in z^4) [6]

(d) Locate the zeros of the function

$$f(z) = z^2(z^2+1)e^z,$$

and find their orders. [3]

(e) Prove that if f is an entire function and

$$f(it) = -t^2 \cos t, \quad \text{for } t > 0,$$

then

$$f(z) = z^2 \cosh z, \quad \text{for } z \in \mathbb{C}.$$ [4]

Question 4 (Unit B4) – 25 marks

(a) Locate the singularities of the function

$$f(z) = \frac{z}{1 - \cos z},$$

and classify each singularity as a removable singularity, a pole (giving its order) or an essential singularity. [10]

(b) Find two Laurent series about 0 for the function

$$f(z) = \frac{5}{(2z-3)(z+1)},$$

one on $\{z : |z| < 1\}$ and the other on $\{z : |z| > 3/2\}$, in each case giving all non-zero terms $a_n z^n$ for $-3 \le n \le 3$. [8]

(c) Let $f(z) = (z + 1/z)\cos(1/z)$.

(i) Find the Laurent series about 0 for f, giving the analytic part and the singular part as far as the term in z^{-3}.

State the annulus of convergence of this Laurent series. [3]

(ii) Determine the nature of the singularity of f at 0. [2]

(iii) Evaluate

$$\int_C (z + 1/z)\cos(1/z)\, dz,$$

where $C = \{z : |z| = 4\}$. [2]